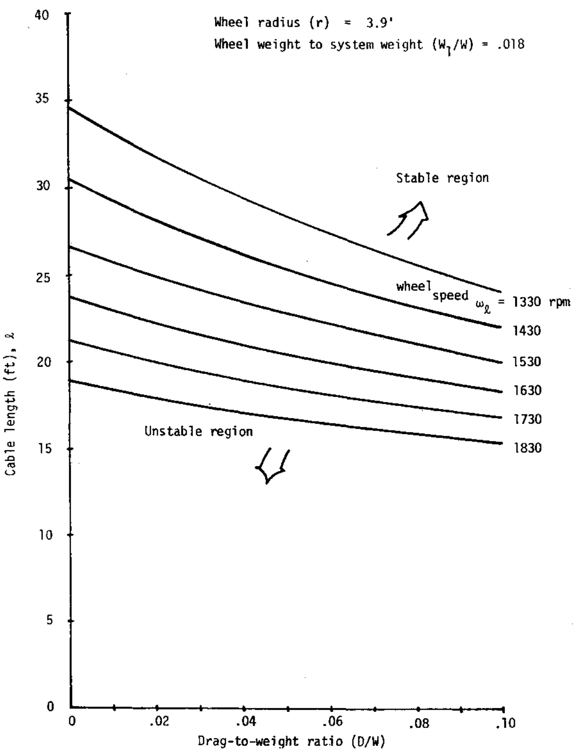

Fig. 5 Lateral stability of system, varying wheel speed.

longitudinal stability in Fig. 2 will not change; however, lateral stability will be greatly affected. Figures 4–6 show the effect that the reaction wheel has on lateral stability. From Fig. 4 it is seen that as the wheel size increases, the cable length required for stability decreases. Similar results are obtained in Figs. 5 and 6 when the wheel speed or weight is increased. These plots show that the cable length can be decreased to almost any length such that lateral stability is no longer the controlling factor, but longitudinal stability is.

While not completely satisfactory, the addition of the reaction wheel is seen to have a tremondous effect on lateral stability. This investigation has given an indication of additional stability parameters, and a possible method of controlling the lateral mode during airborne towing. Further investigations are necessary to determine an effective means of controlling both longitudinal and lateral stability. These investigations are currently underway at the University of Massachusetts.

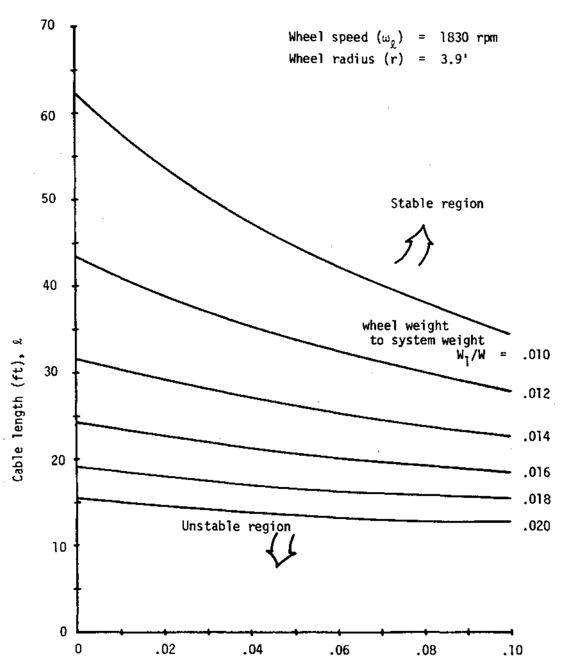

Fig. 6 Lateral stability of system, varying wheel weight.

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On the Fuel Optimality of Cruise

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Introduction

FOR a particular aircraft model where the control variables are thrust and flight path angle, the cruise condition was shown to be a doubly singular arc in the calculus of variations by satisfying the first order necessary conditions. In this note, the singular arc is shown *not* to be minimizing by applying the vector control form of the generalized Legendre-Clebsch condition.

In a paper by Schultz and Zagalsky[1] the minimum fuel-fixed range problem is discussed by considering a particular mathematical model for the aircraft dynamics. In this model thrust and flight path angle are control variables which both assume, within some bounded set, intermediate values during cruise. Cruise is an extremal arc which is called a singular arc in the calculus of variations.[2-4] In Ref. 1 only the first order necessary conditions which generate the singular arc were considered. However, additional second order necessary conditions are available for singular control problems. Here, use is made of the generalized Legendre-Clebsch condition, due first to Kelley,[5] and later generalized to the vector control case by Kelley et al.,[3] Robbins[6] and Goh.[7] It is shown here that the generalized

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vector Legendre-Clebsch condition cannot be satisfied along the cruise arc implying that cruise is not a minimizing arc.

Problem Formulation

In Ref. 1 the minimum fuel for a climb-cruise-descent trajectory is to be found from an initial altitude, range, and velocity to a terminal range, altitude and velocity. Terminal time is not specified although the final answer holds as well for fixed time. The problem is to minimize the fuel

$$f = \int_{t_0}^{t_f} \sigma(E,h)T dt \tag{1}$$

subject to the dynamic constraints

$$\dot{E} = (T - D)V/M; \quad E(t_0) = E_0, \quad E(t_f) = E_f \tag{2}$$
$$\dot{h} = V\gamma; \quad h(t_0) = h_0, \quad h(t_f) = h_f \tag{3}$$
$$\dot{x} = V; \quad x(t_0) = x_0, \quad x(t_f) = x_f \tag{4}$$

where $E \triangleq V^2/2 + gh$ is the specific energy, h is the altitude, V is the velocity, x is the range, $\sigma(E,h)$ is the fuel rate/unit thrust, $D(E,h)$ is the aerodynamic drag, and M is the mass assumed here to be constant.† The control variables are thrust T and flight path angle γ which are bounded as

$$\gamma_{\min} \leq \gamma \leq \gamma_{\max}, \quad T_{\min}(E,h) \leq T \leq T_{\max}(E,h) \tag{5}$$

Define the variational Hamiltonian as

$$H = \sigma T + \lambda_1(T - D)V/M + \lambda_2 V\gamma + \lambda_3 V \tag{6}$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers associated with the differential constraints Eqs. (2-4), respectively. These Lagrange multipliers are propagated as

$$-\dot{\lambda}_1 = (\partial H/\partial E), \quad -\dot{\lambda}_2 = (\partial H/\partial h), \quad -\dot{\lambda}_3 = (\partial H/\partial x) \tag{7}$$

Furthermore, since the Hamiltonian is autonomous and time is free, a first integral of the motion is that

$$H = 0 \tag{8}$$

for all $t \in [t_0, t_f]$.

Analysis of Cruise Condition as Singular Arc

Since the Hamiltonian is linear in the control variables, the maximum principle implies that the controls can only take on intermediate values when their coefficients in the Hamiltonian are zero for a finite interval of time. Otherwise, these coefficients are switching conditions for the control to bang from one bound to the other. Since both T and γ take on intermediate values during cruise, their coefficients are then

$$\partial H/\partial T = \sigma + \lambda_1 V/M = 0 \Rightarrow \lambda_1 = -\sigma M/V \tag{9}$$

$$\partial H/\partial \gamma = \lambda_2 V = 0 \Rightarrow \lambda_2 = 0 \tag{10}$$

over a finite interval of time. Note that the coefficients are explicitly independent of the controls. Therefore, the Legendre-Clebsch necessary condition $\partial^2 H/\partial u^2 \geq 0$ where u is the column vector $u^T = (T,\gamma)$ can only be met with equality. The remaining second order tests and classical sufficiency conditions are not applicable.[2]

†This assumption is used in Ref. 1. The final conclusion remains the same when this restriction is removed.

In Ref. 1 it is shown that when the controls assume values which sustain cruise

$$T = D, \quad \gamma = 0 \tag{11}$$

and since Eq. (8) is satisfied over a finite interval of time, then

$$d(\partial H/\partial T)/dt = \partial[\sigma D/V]/\partial E = 0 \tag{12}$$

$$d(\partial H/\partial \gamma)/dt = \partial[\sigma D/V]/\partial h = 0 \tag{13}$$

Equations (12) and (13) make use of Eq. (8) which implies from Eqs. (10) and (11) that

$$\lambda_3 = -\sigma D/V \tag{14}$$

By cruising at the energy and altitude which satifies $\min(\sigma D/V)$ (which is implied by Eqs. (12) and (13)) and using the controls Eq. (11) the first order necessary conditions are satisfied.

Application of Generalized Legendre-Clebsch Condition

This singular extremal arc is now tested by application of the generalized Legendre-Clebsch condition as given in the following theorem[4] where u is a vector control.

Theorem: A necessary condition for the second variation of the cost to be non-negative for all $u(\cdot)$ belonging to some bound set of controls is that

$$\frac{\partial}{\partial u}\left[\frac{d^q}{dt^q}\frac{\partial H}{\partial u}\right] = 0, \quad \text{for all } t \text{ on the singular arc and } q \text{ odd} \tag{15}$$

and

$$(-1)^p \frac{\partial}{\partial u}\left[\frac{d^{2p}}{dt^{2p}}\frac{\partial H}{\partial u}\right] \geq 0 \text{ for all } t \text{ on the singular arc} \tag{16}$$

For the control vector $u^T = (T,\gamma)$, Eq. (15) is not satisfied since for $q = 1$

$$\begin{bmatrix} \frac{\partial}{\partial T}\left[\frac{d}{dt}\frac{\partial H}{\partial T}\right] & \frac{\partial}{\partial \gamma}\left[\frac{d}{dt}\frac{\partial H}{\partial T}\right] \\ \frac{\partial}{\partial T}\left[\frac{d}{dt}\frac{\partial H}{\partial \gamma}\right] & \frac{\partial}{\partial \gamma}\left[\frac{d}{dt}\frac{\partial H}{\partial \gamma}\right] \end{bmatrix} = \begin{bmatrix} 0 & \left[\frac{\partial \sigma}{\partial h} + \frac{\sigma g}{V^2}\right]V \\ -\left[\frac{\partial \sigma}{\partial h} + \frac{\sigma g}{V^2}\right]V & 0 \end{bmatrix} \tag{17}$$

which is a skew symmetric matrix. This result makes use of Eq. (9). Furthermore, Eq. (17) is true whether or not the time is free or specified. To satisfy the necessary condition Eq. (15) the term $[\partial\sigma/\partial h + \sigma g/V^2]V$ must be zero. Therefore, in general, the cruise condition will not yield a minimizing singular arc.

This result contrary to Ref. 1 is consistent with the energy-state approximation.[8] In the energy state approximation the dynamics are reduced by making V a control variable and eliminating Eq. (3). The variational Hamiltonian is now

$$\bar{H} = \sigma T + \lambda_1(T - D)V/M + \lambda_3 V \tag{18}$$

In the appendix of Ref. 1 it is shown that the velocity set is not convex since the determinate of $\partial^2 H/\partial u^2 = -(\lambda_1/M)^2$ where now the vector $u^T = (T, V)$ and $\partial\bar{H}/\partial T = \sigma + \lambda_1 V/M = 0$ implies that λ_1 is nonzero. Again intermediate values of thrust are not minimizing.

Conclusion

For the minimum fuel-fixed range with time either free or specified, the cruise condition based upon the dynamics of Ref. 1 is found not to be a minimizing singular arc by application of the generalized Legendre-Clebsch condition for vector control. This is seen to be consistent with the results for the energy-state approximation for which intermediate values of thrust are not minimizing. Finally, it can be shown that this conclusion holds when the restriction on constant mass is removed.

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Inviscid Wake-Airfoil Interaction on Multielement High Lift Systems

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ON multielement airfoils, a boundary layer wake is shed by the main airfoil and flows over the flap elements. A viscous interaction of the wake with the boundary layer of the flap element is almost certain. The question arises: does the wake have an inviscid effect on the flap surface pressure, i.e., would the lift of the flap be affected by the wake even if any viscous or apparent turbulent stresses would be absent? A linearized method for estimating the inviscid wake effect and an attempt to explain the effect are presented, and the results are compared with "exact" numerical calculations using a recently developed singularity method.[1]

According to Küchemann[2] and Thwaites[3] the lift of the flap is affected by the wake in inviscid fluid even for cases where potential flow is present between the wake and the flap. As a qualitative answer, Refs. 2 and 3 mention that the additional lift force on airfoils induced by nonuniformities in the (inviscid) freestream flow always tends to be directed toward the region with the highest velocity or total pressure. In the case of a wake flowing over the upper surface of a flap, the lift of the flap would be reduced by inviscid interaction.

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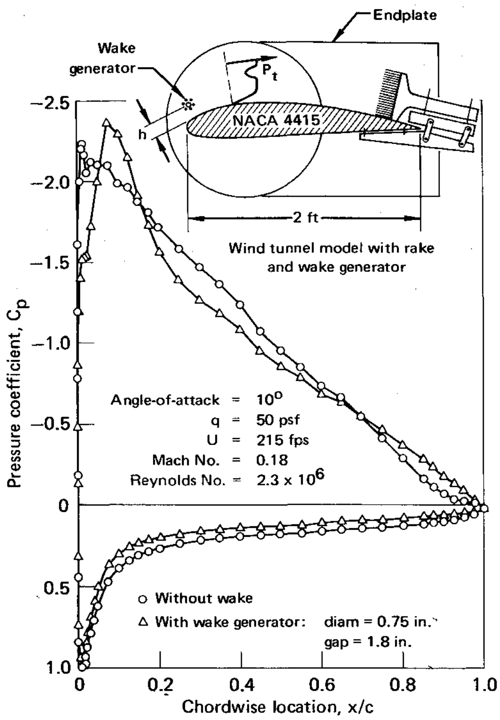

Fig. 1 Experimental airfoil surface pressure distributions with and without wake.[5]

Experimental verification of the inviscid interaction is difficult because viscosity and turbulence effects are always superimposed onto the inviscid effects. Ljungström[4] conducted experiments on a 2-D airfoil model equipped with slat and flap. By different means, he was able to vary the momentum defect of the wake leaving the main wing trailing edge. The measurements of flap surface pressure distributions for different wake momentum defects on otherwise identical configurations provide an indication of the inviscid wake effect. The results show less negative static pressure on the flap upper surface for the larger wake and almost no difference on the flap lower surface, in agreement with the present inviscid calculations. Pressure distributions obtained in recent experiments by the authors[5] are shown in Fig. 1. Here the wake was generated by a bundle of transverse rods near the leading edge of a NACA 4415 section. The difference of the surface pressures measured near midchord with and without the wake is approximately as expected for the inviscid wake interaction; however, other effects of the viscous wake must also be considered.

For further discussion, a simplified flow model is defined as follows: one airfoil (representing the flap) is placed in a two-dimensional, incompressible, inviscid flow which is irrotational except in the region of the wake. The wake is described by total pressure profiles, and it originates at upstream infinity or at a "wake generator" defined below. For this type flow the equations of motion relate the wake total pressure to the local value of the stream function in a unique way for the entire wake. The wake generator in this analysis is a straight line segment in the plane of flow of length, h. The total pressure, H, of fluid crossing this line segment is instantly reduced by ΔH; however, the flow velocity is continuous across the wake generator. The resulting total pressure wake profiles have rectangular shapes. The force on the wake generator, $F = h\Delta H$, is perpendicular to the line segment. The com-